Depth-Optimal Synthesis of Clifford Circuits with SAT Solvers

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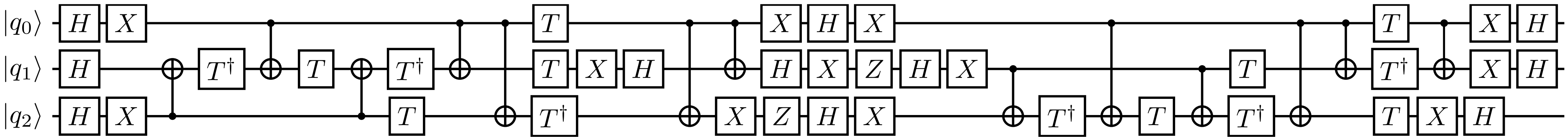
https://www.cda.cit.tum.de/research/quantum/

Depth-Optimal Synthesis of Quantum Circuits

- Given a quantum circuit



- How to synthesize a depth-optimal version of this?

Classical Equivalence Checking

- Two classical n-bit circuits C, C' are said to be equivalent if and only of the agree on all inputs:

$$\forall x \in \{0,1\}^n\colon\ Cx = C'x$$

  - Negated:

$$\exists x \in \{0,1\}^n\colon\ Cx \neq C'x$$

- Can be expressed as a Boolean formula $\Phi_{C,C'}(x)$
  - Equivalent to SAT (**NP Complete**)

Classical (Depth-Optimal) Synthesis

- Given a classical circuit C, find an equivalent circuit with maximal depth d_{max} :

$$\exists C_d, depth(C_d) \leq d_{max}\colon \forall x \in \{0,1\}^n\colon\ C_d x = Cx$$

- Since any logical circuit can be represented by binary encoding of size at most $poly(n\ d_{max})$:

$$\exists y \in \{0,1\}^{poly(n\ d_{max})}\colon \forall x \in \{0,1\}^n : \phi_C(y,x) = 1$$

- The problem of solving this QBF is **complete for $\boldsymbol{\Sigma_2^p}$**

Going Quantum

- **Equivalence Checking**: Two n-qubit quantum circuits U, V are equivalent if for all input states:

$$\forall|\psi\rangle \in \mathbb{C}^{2^n}: U|\psi\rangle = e^{i\phi}V|\psi\rangle$$

- Theoretically infinitely many input states need to be checked
  - Can be reduced to just 4^n but still exponential
- The (negated) quantum circuit equivalence checking problem is **QMA complete**

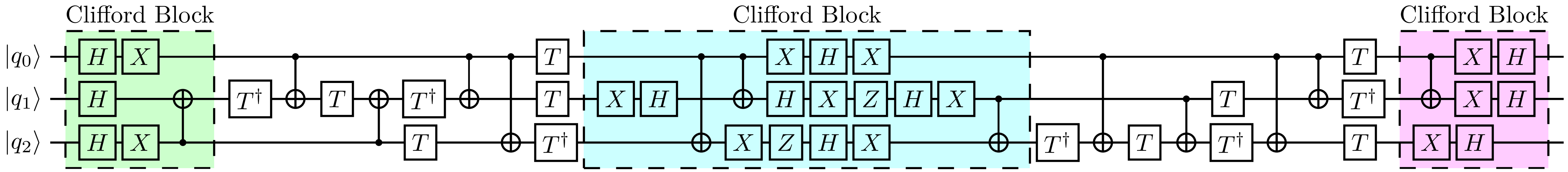
- **Circuit Synthesis**:

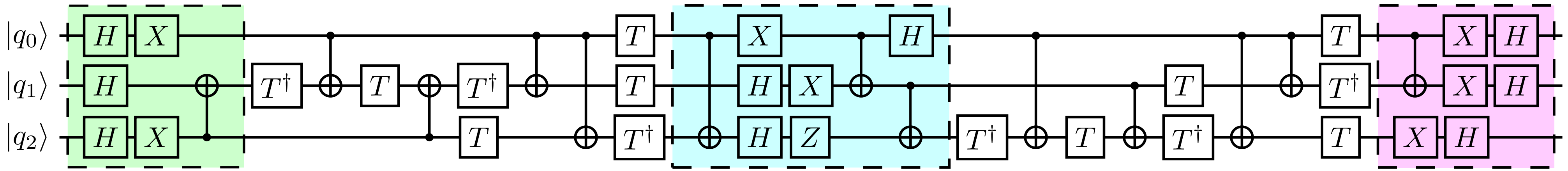
$$\exists U_d, depth(U_d) \leq d_{max}: \forall|\psi\rangle: U_d|\psi\rangle = e^{i\phi}U|\psi\rangle$$

$$\exists U_d, depth(U_d) \leq d_{max}: \forall|\psi\rangle: U^\dagger U_d|\psi\rangle = e^{i\phi}|\psi\rangle$$

- At least as hard as classical synthesis

Clifford Optimization





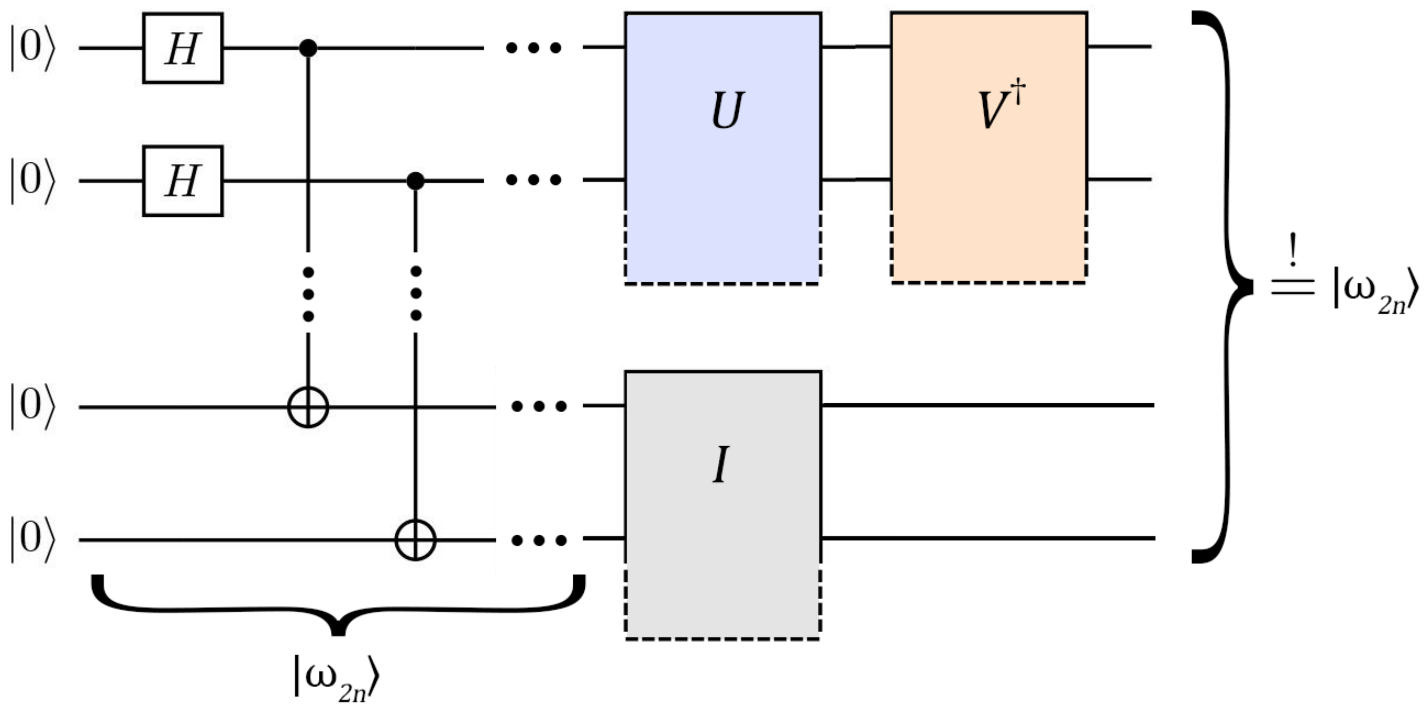
Entanglement-Assisted Equivalence Checking

- Equivalence Checking:

$$\forall|\psi\rangle \in \mathbb{C}^{2^n}\ V^\dagger U|\psi\rangle = e^{i\phi}|\psi\rangle$$

- Synthesis:

$$\exists U_d, depth(U_d) \leq d_{max}\colon \forall|\psi\rangle\ U^\dagger U_d|\psi\rangle = e^{i\phi}|\psi\rangle$$



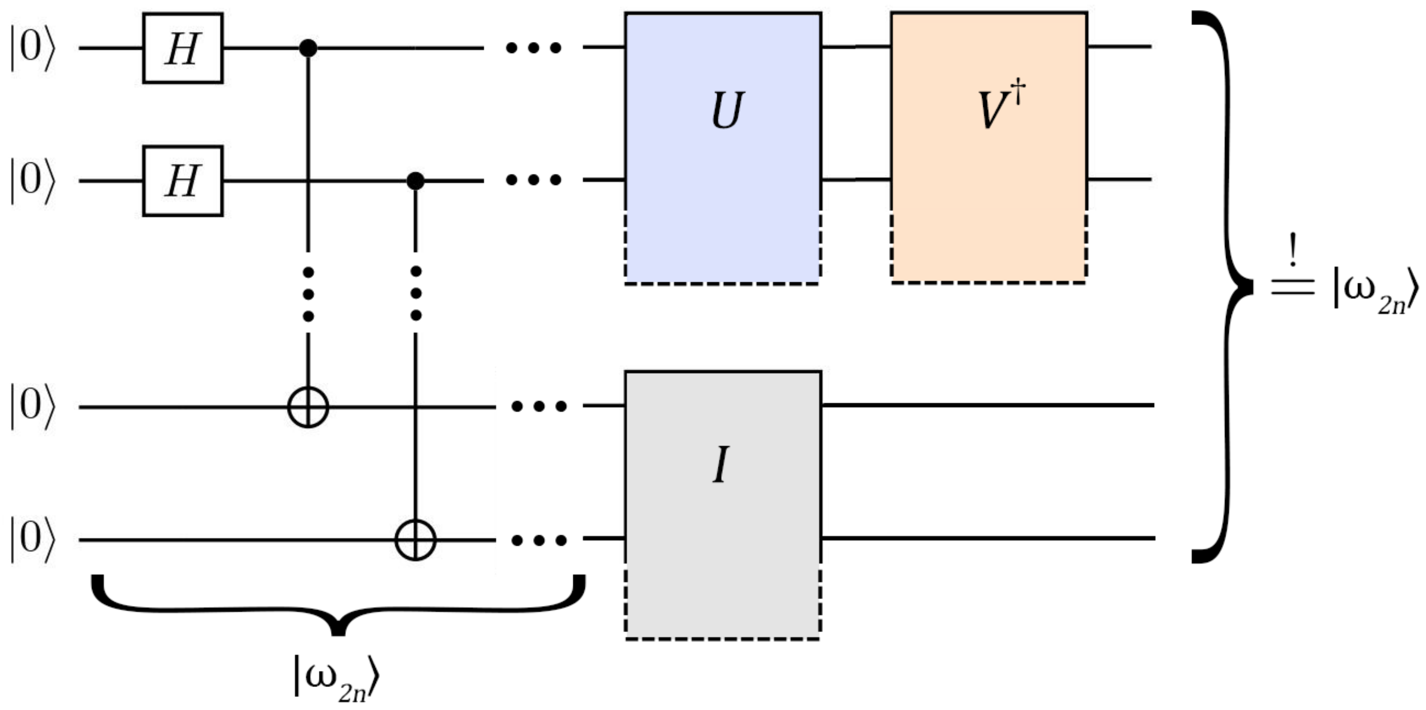
Entanglement-Assisted Equivalence Checking

- Equivalence Checking:

$$(V^{\dagger}U) \otimes I |\omega_{2n}\rangle\langle\omega_{2n}| \left(V^{\dagger}U\right)^{\dagger} \otimes I = |\omega_{2n}\rangle\langle\omega_{2n}|$$

- Synthesis:

$$\exists U_d, depth(U_d) \leq d_{max}: \forall |\psi\rangle\ U^{\dagger}U_d|\psi\rangle = e^{i\phi}|\psi\rangle$$



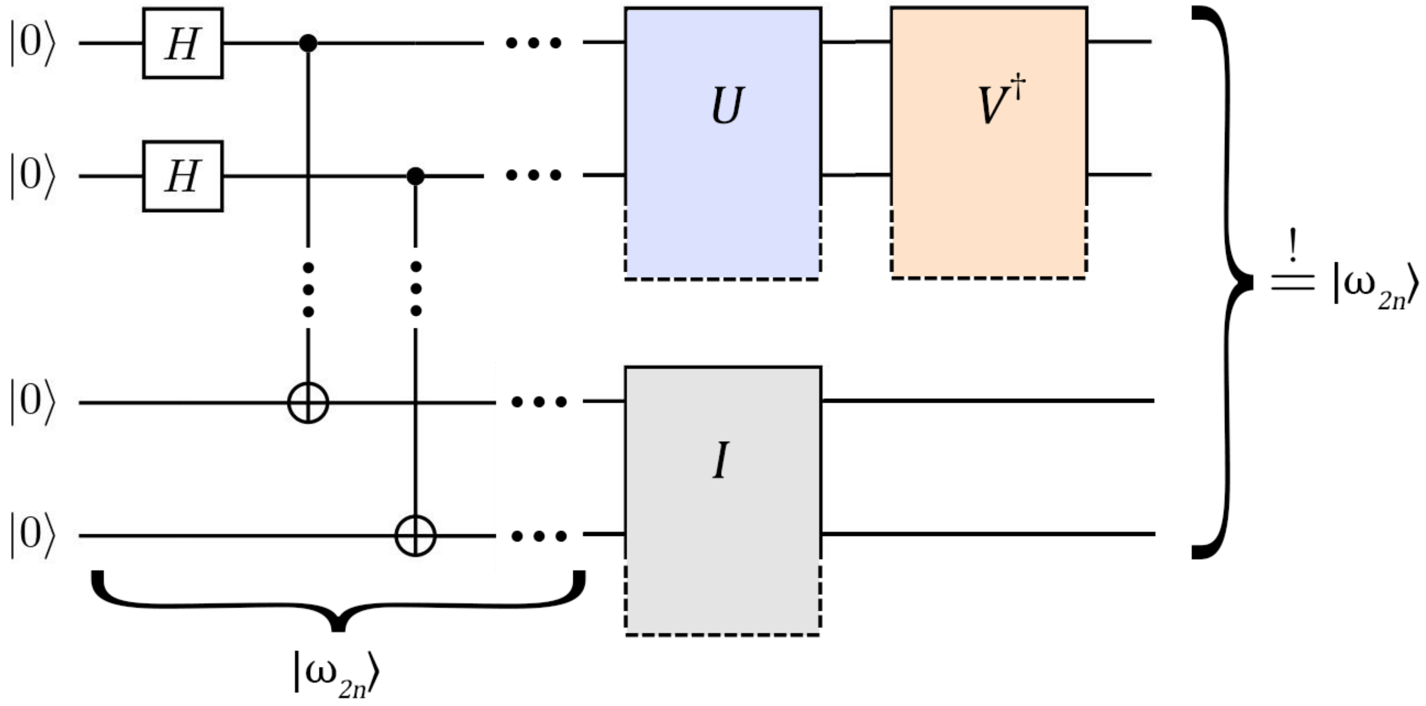
Entanglement-Assisted Equivalence Checking

- Equivalence Checking:

$$(V^\dagger U) \otimes I|\omega_{2n}\rangle\langle\omega_{2n}|\left(V^\dagger U\right)^\dagger \otimes I = |\omega_{2n}\rangle\langle\omega_{2n}|$$

- Synthesis:

$$\exists U_d, depth(U_d) \leq d_{max}: (U_d^\dagger U) \otimes I|\omega_{2n}\rangle\langle\omega_{2n}|\left(U_d^\dagger U\right)^\dagger \otimes I = |\omega_{2n}\rangle\langle\omega_{2n}|$$



Efficient Encoding

$$\exists U_d, depth(U_d) \leq d_{max} (U_d^\dagger U) \otimes I |\omega_{2n}\rangle\langle\omega_{2n}| \left(U_d^\dagger U\right)^\dagger \otimes I = |\omega_{2n}\rangle\langle\omega_{2n}|$$

- Efficient Simulation ⟹ Efficient Encoding into CNF
- In general hard
- For Clifford Circuits simulation is polynomial
  - We will explicitly construct a polynomial SAT encoding of the above formula

Theorem

Let U be a n-qubit Clifford unitary and fix a maximum depth $d_{max} \in \mathbb{N}$. Then, the decision problem "*is it possible to exactly reproduce U with (at most) d_{max} Clifford layers?*" can be rephrased as an instance of SAT with $O(n^2 d_{max})$ variables and $O(n^4 d_{max})$ clauses of constant size each.

The Gottesman-Knill Theorem

- Clifford circuits can be efficiently simulated on classical computers
- Any n-qubit stabilizer state can be represented by

  $$\pm P_{i,0} P_{i,1} \dots P_{i,n-1}$$

  with

  $$P_{i,j} \in \{I, X, Y, Z\} \quad and \quad i, j = 0,1, \dots, n-1$$

  which describe the generators for the group of stabilizers
- Tableau representation

$$\left[\begin{array}{ccc|ccc|c} x_{0,0} & \cdots & x_{0,n-1} & z_{0,0} & \cdots & z_{0,n-1} & r_0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1,0} & \cdots & x_{n-1,n-1} & z_{n-1,0} & \cdots & z_{n-1,n-1} & r_{n-1} \end{array}\right] = [\boldsymbol{x}_0 \quad \cdots \quad \boldsymbol{x}_{n-1} | \quad \boldsymbol{z}_0 \quad \cdots \quad \boldsymbol{z}_{n-1} | \quad \boldsymbol{r}]$$

- Memory: $n(2n+1)$ bits (compared to 2^n complex amplitudes)
  - For the 2n-qubit state $|\omega_{2n}\rangle$ the tableau would require $2n(4n+1)$ bits
  - This can be reduced to $2n(2n+1)$ bits by considering stabilizers and *destabilizers*

Gate-Time Encoding

Tableau at depth d

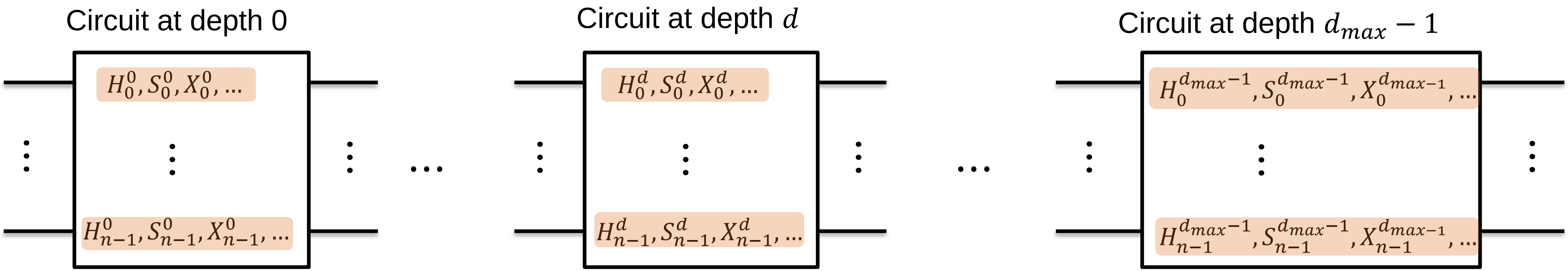
$$\left[\boldsymbol{x_0^d} \cdots \boldsymbol{x_{n-1}^d} \middle| \boldsymbol{z_0^d} \cdots \boldsymbol{z_{n-1}^d} \middle| \boldsymbol{r^d}\right]$$

$H_i^d \Rightarrow \boldsymbol{z_i^{d+1}} = \boldsymbol{x_i} \wedge \boldsymbol{x_i^{d+1}} = \boldsymbol{z_i^d}$

Tableau at depth $d+1$

$$\left[\boldsymbol{x_0^{d+1}} \cdots \boldsymbol{x_{n-1}^{d+1}} \middle| \boldsymbol{z_0^{d+1}} \cdots \boldsymbol{z_{n-1}^{d+1}} \middle| \boldsymbol{r^{d+1}}\right]$$

Consistency Constraint: Exactly one gate per qubit per depth



- Gate variables:

$$Svars = \{g_q^d \mid g \in SQGs, q \in Q, 0 \leq d < d_{max}\}$$

$$Tvars = \{g_{q_0,q_1}^d \mid g \in TQGs, q_0 \in Q, q_1 \in Q \ / \{q_0\}\}$$

- Tableau variables:

$$Xvars = \{x_q^d \mid q \in Q, 0 \leq d < d_{max}\}$$

$$Zvars = \{z_q^d \mid q \in Q, 0 \leq d < d_{max}\}$$

$$Rvars = \{r^d \mid 0 \leq d < d_{max}\}$$

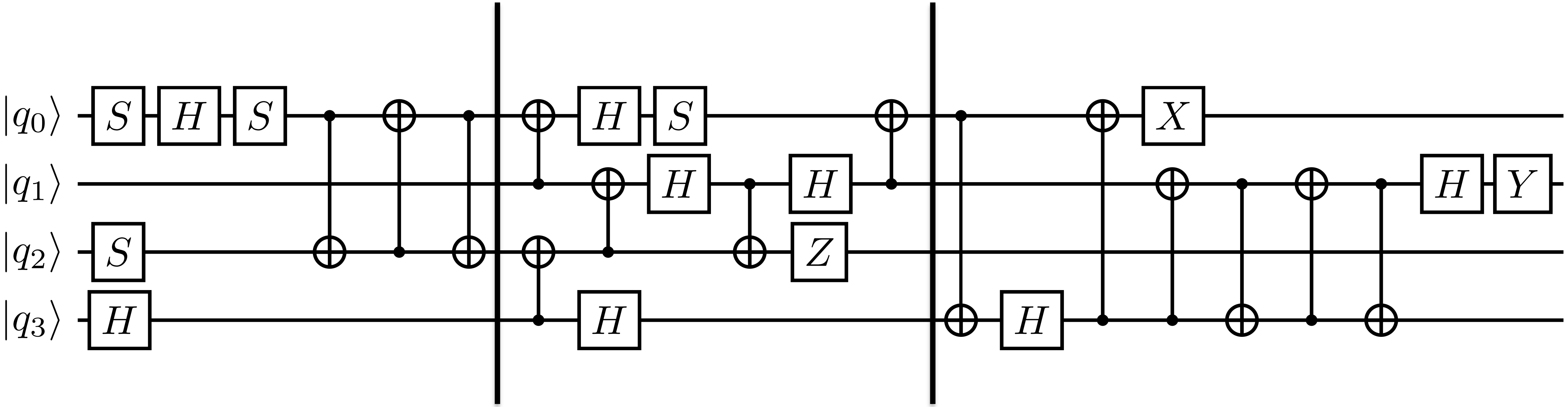
Experimental Results for Exact Synthesis

	Optimal			Bravyi et al.		
n	d	$\|G\|$	t [s]	d	$\|G\|$	t [s]
3	5.70	11.40	0.33	11.70	16.10	0.18
4	6.60	16.70	4.10	16.00	23.50	0.16
5	7.60	25.00	381.95	22.90	37.30	0.18
6	-	-	-	29.40	55.10	0.20
7	-	-	-	37.00	70.20	0.17
8	-	-	-	42.10	86.30	0.20
9	-	-	-	53.50	108.80	0.28
10	-	-	-	59.90	128.70	0.28
11	-	-	-	72.20	157.30	0.25
12	-	-	-	78.90	170.10	0.26
13	-	-	-	91.40	207.10	0.34
14	-	-	-	100.30	235.20	0.33

n: Number of qubits d: Average depth $|G|$: Average gate count t: average runtime

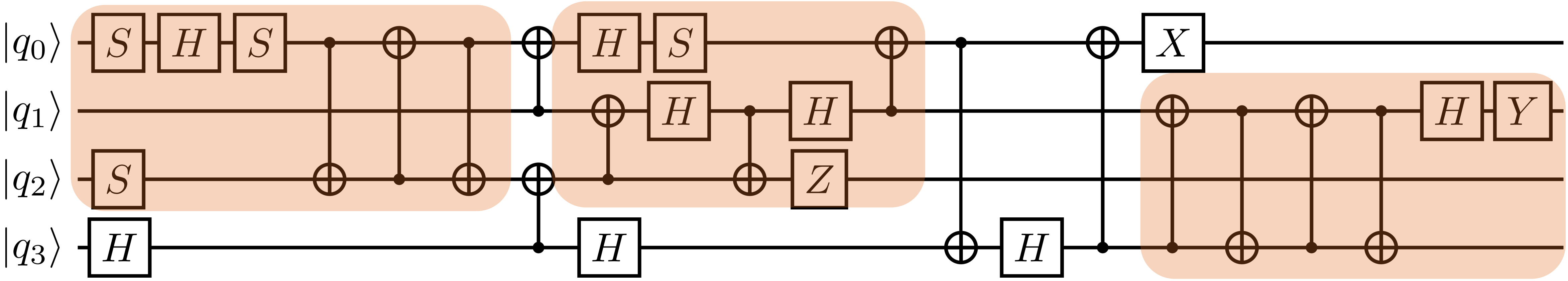
- 10 random Clifford circuits per number of Qubits
- Results averaged over all optimized circuits

Heuristic Version 1: Split Circuit Vertically



- Split circuit vertically into independent subcircuits
- Synthesize optimally in parallel

Heuristic Version 2: Split Circuit Horizontally



- Identify subcircuits involving less qubits
- Synthesize optimally in parallel

Experimental Results for Heuristic Synthesis

	Optimal			Heuristic Vertical			Heuristic Horizontal			Bravyi et al.		
n	d	$\|G\|$	t [s]	d	$\|G\|$	t [s]	d	$\|G\|$	t [s]	d	$\|G\|$	t [s]
3	5.70	11.40	0.33							11.70	16.10	0.18
4	6.60	16.70	4.10							16.00	23.50	0.16
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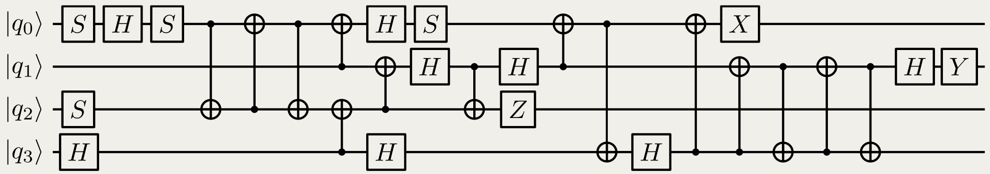
MQT QMAP

clifford.py

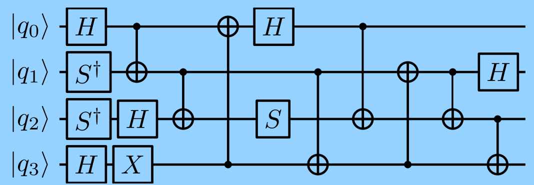
```python
from mqt import qmap

qc =
```




```python
# depth = 21

qc_opt, res = qmap.optimize_clifford(qc,
                                     include_destabilizers=True,
                                     target_metric='depth')


qc_opt.draw()
print(res.depth)
```

Terminal

```
$ python3 clifford.py

 7
```

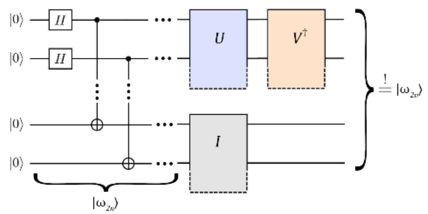



```
pip install mqt.qmap
```

Conclusion

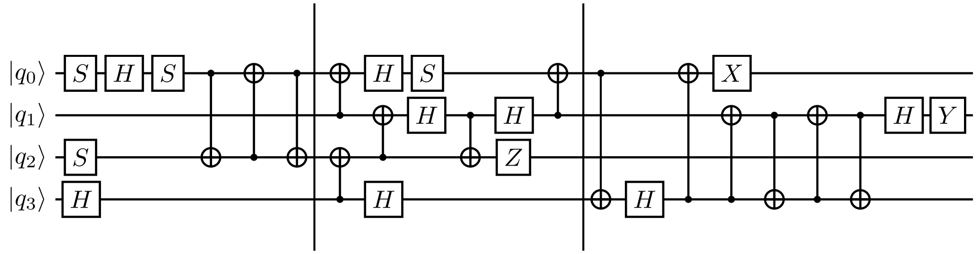
- Clifford Circuit Synthesis is at most as hard as classical equivalence checking

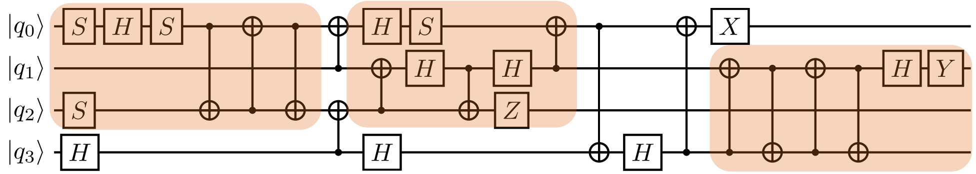
$$\exists U_d, depth(U_d) \leq d_{max} \; \forall|\psi\rangle \; U^\dagger U_d|\psi\rangle = e^{i\phi}|\psi\rangle$$



$$\exists U_d, depth(U_d) \leq d_{max} (U_d^\dagger U) \otimes I|\omega_{2n}\rangle\langle\omega_{2n}|\left(U_d^\dagger U\right)^\dagger \otimes I = |\omega_{2n}\rangle\langle\omega_{2n}|$$

- Synthesis can be formulated as SAT problem
- SAT formulation can give **provably** depth-optimal Clifford circuits
- Better scaling with heuristic





https://github.com/cda-tum/mqt-qmap

- Implementation publicly available